Chapter 1

Funkce vice promennych - uvod

Priklad 1.1. *Urcete a nakreslete definicni obor, obor hodnot a vrstevnice nasledujicich funkci. Pro* $c \in R_f$ *definujeme vrstevnici funkce* f *odpovidajici hodnote* c *nasledovne:*

$$M_c := \{[x, y] \in \mathbb{R}^2;\ f(x, y) = c\} = f^{-1}(\{c\}).$$

(a) $f(x, y) := x + \sqrt{y}$*:*

$$\begin{aligned} D_f &= \{[x, y] \in \mathbb{R}^2;\ y \geq 0\}, \\ R_f &= \mathbb{R}, \\ M_c &= \{[x, y] \in \mathbb{R}^2;\ y = (c - x)^2 \wedge c \geq x\},\ c \in R_f. \end{aligned}$$

(b) $f(x, y) := \frac{y}{x}$*:*

$$\begin{aligned} D_f &= \{[x, y] \in \mathbb{R}^2;\ x \neq 0\}, \\ R_f &= \mathbb{R}, \\ M_c &= \{[x, y] \in \mathbb{R}^2;\ y = cx\},\ c \in R_f. \end{aligned}$$

(c) $f(x, y) := x^2 + y^2$*:*

$$\begin{aligned} D_f &= \mathbb{R}^2, \\ R_f &=< 0, +\infty), \\ M_c &= S(\boldsymbol{o}, \sqrt{c}), \textit{kruznice o stredu } \boldsymbol{o} \textit{ a polomeru } \sqrt{c},\ c \in R_f. \end{aligned}$$

(d) $f(x, y) := x^2 - y^2$*:*

$$\begin{aligned} D_f &= \mathbb{R}^2, \\ R_f &= \mathbb{R}, \\ M_c &: \ \textit{hyperboly},\ c \in R_f. \end{aligned}$$

(e) $f(x, y) := \sqrt{xy}$*:*

$$\begin{aligned} D_f &= \{[x, y] \in \mathbb{R}^2;\ xy \geq 0\}, \\ R_f &=< 0, +\infty), \\ M_c &= \left\{[x, y] \in \mathbb{R}^2;\ y = \frac{c^2}{x}\right\},\ c > 0, \\ M_0 &= \left\{[x, y] \in \mathbb{R}^2;\ xy = 0\right\}, \textit{osy } x \textit{ a } y. \end{aligned}$$

(f) $f(x,y) := \sqrt{1-x^2-y^2}$*:*

$$D_f = \overline{B(\boldsymbol{o},1)},$$
$$R_f = <0,1>,$$
$$M_c = S(\boldsymbol{o},\sqrt{1-c^2}), \textit{kruznice o stredu } \boldsymbol{o} \textit{ a polomeru } \sqrt{1-c^2},\ c \in R_f.$$

(g) $f(x,y) := \frac{1}{\sqrt{x^2+y^2-1}}$*:*

$$D_f = (\overline{B(\boldsymbol{o},1)})^C,$$
$$R_f = (0,+\infty),$$
$$M_c = S\left(\boldsymbol{o},\sqrt{c^{-2}+1}\right), \textit{kruznice o stredu } \boldsymbol{o} \textit{ a polomeru } \sqrt{c^{-2}+1},\ c \in R_f.$$

(h) $f(x,y) := \sqrt{(x^2+y^2-1)(4-x^2-y^2)}$*:*

$$D_f = \overline{B(\boldsymbol{o},2)} \setminus B(\boldsymbol{o},1),$$
$$R_f = \left\langle 0, \frac{3}{2} \right\rangle,$$
$$M_c = S\left(\boldsymbol{o},\sqrt{\frac{5+\sqrt{9-4c^2}}{2}}\right) \cup S\left(\boldsymbol{o},\sqrt{\frac{5-\sqrt{9-4c^2}}{2}}\right),\ c \in R_f.$$

(i) $f(x,y) := \sqrt{1-(x^2+y)^2}$*:*

$$D_f = \{[x,y] \in \mathbb{R}^2;\ -1-x^2 \leq y \leq 1-x^2\},$$
$$R_f = <0,1>,$$
$$M_c = \left\{[x,y] \in \mathbb{R}^2;\ y = \pm\sqrt{1-c^2}-x^2\right\},\ c \in R_f.$$

(j) $f(x,y) := \sqrt{\sin(x^2+y^2)}$*:*

$$D_f = \bigcup_{n=0}^{\infty} \left(\overline{B\left(\boldsymbol{o},\sqrt{(2n+1)\pi}\right)} \setminus B\left(\boldsymbol{o},\sqrt{2n\pi}\right)\right),$$
$$R_f = <0,1>,$$
$$M_c = \bigcup_{n=0}^{\infty} \left(S\left(\boldsymbol{o},\sqrt{2n\pi+\arcsin(c^2)}\right) \cup S\left(\boldsymbol{o},\sqrt{(2n+1)\pi-\arcsin(c^2)}\right)\right),\ c \in R_f.$$

(k) $f(x,y) := \operatorname{sign}(\sin(x)\sin(y))$*:*

$$D_f = \mathbb{R}^2,$$
$$R_f = \{-1,0,1\},$$
$$M_c:\ \textit{sachovnice},\ c \in R_f.$$

(l) $f(x,y) := |x|+y$*:*

$$D_f = \mathbb{R}^2,$$
$$R_f = \mathbb{R},$$
$$M_c = \{[x,y] \in \mathbb{R}^2;\ y = c-|x|\},\ c \in R_f.$$

Priklad 1.2. *Naleznete definicni obor a rozhodnete o spojitosti funkce* $\frac{\cos(xy)}{\sqrt{x^2+y^2}}$.

Poznamka 1.3. *Funkce* $f(x,y) := \rho_e(x,y) : \mathbb{R}^{2n} \to \mathbb{R}$ *je spojita.*

Proof. Z definice metriky plyne, ze

$$\rho_e([\boldsymbol{x}_0, \boldsymbol{y}_0], [\boldsymbol{x}_1, \boldsymbol{y}_1]) \geq \max\{\rho_e(\boldsymbol{x}_0, \boldsymbol{x}_1), \rho_e(\boldsymbol{y}_0, \boldsymbol{y}_1)\}.$$

Pak dvojim pouzitim trojuhelnikove nerovnosti obdrzime

$$\begin{aligned} |f(\boldsymbol{x}_0, \boldsymbol{y}_0) - f(\boldsymbol{x}_1, \boldsymbol{y}_1)| &= |\rho_e(\boldsymbol{x}_0, \boldsymbol{y}_0) - \rho_e(\boldsymbol{x}_1, \boldsymbol{y}_1)| \\ &\leq |\rho_e(\boldsymbol{x}_0, \boldsymbol{y}_0) - \rho_e(\boldsymbol{x}_0, \boldsymbol{y}_1)| + |\rho_e(\boldsymbol{x}_0, \boldsymbol{y}_1) - \rho_e(\boldsymbol{x}_1, \boldsymbol{y}_1)| \\ &\leq \rho_e(\boldsymbol{y}_0, \boldsymbol{y}_1) + \rho_e(\boldsymbol{x}_0, \boldsymbol{x}_1) \leq 2\rho_e([\boldsymbol{x}_0, \boldsymbol{y}_0], [\boldsymbol{x}_1, \boldsymbol{y}_1]), \end{aligned}$$

z cehoz plyne, ze f je spojita.

□

Priklad 1.4. *Necht* $n \in \mathbb{N}$ *a funkce* $f, g : \mathbb{R}^n \to \mathbb{R}$ *jsou spojite v bode* $a \in \mathbb{R}$. *Pak funkce* $F(x) := \max\{f(x), g(x)\}$ *je rovnez spojita v bode* a.

Proof. Bud $\varepsilon > 0$ libovolne, hledame $\delta > 0$ takove, ze $|F(x) - F(a)| < \varepsilon$ pro vsechna $x \in B(a, \delta)$. Pokud $f(a) > g(a)$, tak nalezneme (ze spojitosti f a g) $\delta_0, \delta_1, \delta_2 > 0$, ze

$$\begin{aligned} \forall x \in B(a, \delta_0) &: |f(x) - f(a)| < \varepsilon, \\ \forall x \in B(a, \delta_1) &: f(x) > \frac{f(a) + g(a)}{2}, \\ \forall x \in B(a, \delta_2) &: g(x) < \frac{f(a) + g(a)}{2}. \end{aligned}$$

Polozime $\delta := \min\{\delta_0, \delta_1, \delta_2\}$ a jsme hotovi. Pripad $f(a) < g(a)$ se dokaze obdobne. Pokud $f(a) = g(a)$, tak nalezneme (ze spojitosti f a g) $\delta_0, \delta_1 > 0$, ze

$$\begin{aligned} \forall x \in B(a, \delta_0) &: |f(x) - f(a)| < \varepsilon, \\ \forall x \in B(a, \delta_1) &: |g(x) - g(a)| < \varepsilon. \end{aligned}$$

Polozime $\delta := \min\{\delta_0, \delta_1\}$ a jsme hotovi.

□

Priklad 1.5. *S pouzitim V11 (a take V2, V5) rozhodnete, zda nasledujici mnoziny jsou otevrene, ci uzavrene.*

(a) $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2 + \sin(xyz^3) > \cos(y^2)\}$*: otevrena,*

(b) $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2 + \arctan(xe^y z^3) \leq \sin(y^2)\}$*: uzavrena,*

(c) $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2 + \cos(x^2yz^3) = 5 + e^{y^2}\}$*: uzavrena,*

(d) $M := \{[x,y,z] \in \mathbb{R}^3;\ x^2 > \sin(xy^2) \wedge (y^2 < \cos(z) \vee x < y)\}$*: otevrena.*

Poznamka 1.6. *Mnozina*

$$\begin{aligned} M := \Bigg\{ [x,y,z] \in \mathbb{R}^3;\ \sin\left(xe^{x^2y}\right) > \cos\left(\arctan(z)y^2\right) \\ \wedge \left(\exists k \in \mathbb{N}:\ \left|x^2 + xy^2z^3\right| \in \left(k, k + \frac{1}{k}\right)\right) \Bigg\} \end{aligned}$$

je otevrena.

Proof. Polozime

$$A := \left\{ [x,y,z] \in \mathbb{R}^3;\ \sin\left(xe^{x^2y}\right) > \cos\left(\arctan(z)y^2\right) \right\},$$
$$B := \left\{ [x,y,z] \in \mathbb{R}^3;\ \exists k \in \mathbb{N}:\ \left|x^2 + xy^2z^3\right| \in \left(k, k+\frac{1}{k}\right) \right\},$$
$$B_k := \left\{ [x,y,z] \in \mathbb{R}^3;\ \left|x^2 + xy^2z^3\right| \in \left(k, k+\frac{1}{k}\right) \right\},\ k \in \mathbb{N},$$
$$B_k^1 := \left\{ [x,y,z] \in \mathbb{R}^3;\ \left|x^2 + xy^2z^3\right| - k > 0 \right\},\ k \in \mathbb{N},$$
$$B_k^2 := \left\{ [x,y,z] \in \mathbb{R}^3;\ \left|x^2 + xy^2z^3\right| - k - \frac{1}{k} < 0 \right\},\ k \in \mathbb{N}.$$

Pak dle (V11) jsou mnoziny B_k^1 a B_k^2 otevrene a tedy dle ($V2$) jsou otevrene i mnoziny $B_k = B_k^1 \cap B_k^2$. Z (V2) a $B = \bigcup_{k=1}^{\infty} B_k$ plyne, ze mnozina B je rovnez otevrena. Otevrenost mnoziny M plyne opet z (V2) a $M = A \cap B$.

□

1.0.1 limity funkci vice promennych

Postup reseni $\lim_{[x,y]\to[x_0,y_0]} f(x,y)$:

Nejprve se rozhodneme, zda budeme zkouset dokazovat existenci ci neexistenci. V pripade, ze nam jedno nepujde, tak muzeme zkusit dokazovat to druhe.

Jeden z moznych zpusobu jak se pokusit dokazat neexistenci: Nalezneme spojite proste krivky (vetsinou primky) ϕ_1 a ϕ_2 takove, ze $\phi_i(0) = [x_0, y_0]$ a $\lim_{t\to 0} f(\phi_1(t)) \neq \lim_{t\to 0} f(\phi_2(t))$. Pokud by limita existovala, tak z vety o limite slozene funkce plyne, ze $\lim_{t\to 0} f(\phi_i(t)) = \lim_{[x,y]\to[x_0,y_0]} f(x,y)$.

Pokud dokazujeme existenci, tak je dobre nejprve nalezt vhodneho kandidata na limitu (napr. spocitat limitu po nejake krivce viz. predchozi odstavec). Pokud jiz takoveho kandidata mam (predpokladejme, ze je vlastni), pak jeden ze zpusobu jak dokazat, ze $\lim_{[x,y]\to[x_0,y_0]} f(x,y) = A \in \mathbb{R}$ je nalezeni vhodne funkce $g(x,y)$ takove, ze $|f(x,y) - A| \leq g(x,y)$ na prstencovem okoli $[x_0, y_0]$ a pro kterou vime, ze $\lim_{[x,y]\to[x_0,y_0]} g(x,y) = 0$ (napriklad je spojita v bode $[x_0, y_0]$ a $g(x_0, y_0) = 0$). V pripade nevlastnich limit (napr. $+\infty$) lze postupovat obdobne. Hledame "jednoduchou" funkci g, ze $f(x,y) \geq g(x,y)$ na prstencovem okoli $[x_0, y_0]$ a $\lim_{[x,y]\to[x_0,y_0]} g(x,y) = +\infty$. Pro nalezeni vhodneho odhadu (funkci g) lze mnohdy vyuzit mocnine nerovnosti (napr. AG nerovnost).

Priklad 1.7. *Rozhodnete, zda existuji nasledujici limity a pokud ano, tak je spoctete.*

(i) $\lim_{[x,y]\to[0,0]} \frac{x-y}{x+y}$. *Ne:* $x = 0$, $y = 0$.

(ii) $\lim_{[x,y]\to[0,0]} \frac{xy}{x^2+y^2}$. *Ne:* $x = 0$, $y = x$.

(iii) $\lim_{[x,y]\to[0,0]} \frac{x^2y}{x^2+y^2}$. *Ano* 0*:* $x^2 + y^2 \geq 2xy$.

(iv) $\lim_{[x,y]\to[0,0]} \frac{x^2y}{x^4+y^2}$. *Ne: Nepomohou primky, je treba pouzit* $x = 0$, $y = x^2$.

(v) $\lim_{[x,y]\to[0,0]} (x+y)^2 \sin\left(\frac{1}{\sqrt{x^2+y^2}}\right)$. *Ano* 0*: Nulova krat omezena.*

(vi) $\lim_{[x,y]\to[0,0]} \frac{(x+y)^2}{\sin(\sqrt{x^2+y^2})}$. *Ano* 0*:*

$$
\begin{aligned}
g(x,y) :=& \frac{(|x|+|y|)^2}{\sin\left(\frac{|x|+|y|}{\sqrt{2}}\right)} \geq \quad |f(x,y)|,\ t := |x|+|y| \neq 0,\ [x,y] \neq [0,0],\\
\varphi(t) :=& \frac{t^2}{\sin\left(\frac{t}{\sqrt{2}}\right)} = \qquad g(x,y),\ \lim_{[x,y]\to[0,0]} |x|+|y| = 0,\\
\lim_{t\to 0}\varphi(t) =& 0.
\end{aligned}
$$